Mathematics: analysis and approaches
Standard Level
Paper 1

WORKED SOLUTIONS

Date: ____________________

1 hour 30 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

exam: 9 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

Let $f(x) = \cos 4x$ and $g(x) = e^{3x-1}$

(a) Find $f'(x)$. [2]

(b) Find $g'(x)$. [2]

(c) Let $h(x) = g(x) \times f(x)$. Find $h'(x)$. [2]

(a) Applying the chain rule:

$$f'(x) = (-\sin 4x)4$$

$$f'(x) = -4\sin 4x$$

(b) Applying the chain rule:

$$g'(x) = \left(e^{3x-1}\right)3$$

$$g'(x) = 3e^{3x-1}$$

(c) $h(x) = \left(e^{3x-1}\right)(\cos 4x)$

Applying the product rule:

$$h'(x) = \left(3e^{3x-1}\right)(\cos 4x) + \left(e^{3x-1}\right)(-4\sin 4x)$$

$$h'(x) = 3e^{3x-1}\cos 4x - 4e^{3x-1}\sin 4x \quad \text{or} \quad h'(x) = e^{3x-1}(3\cos 4x - 4\sin 4x)$$

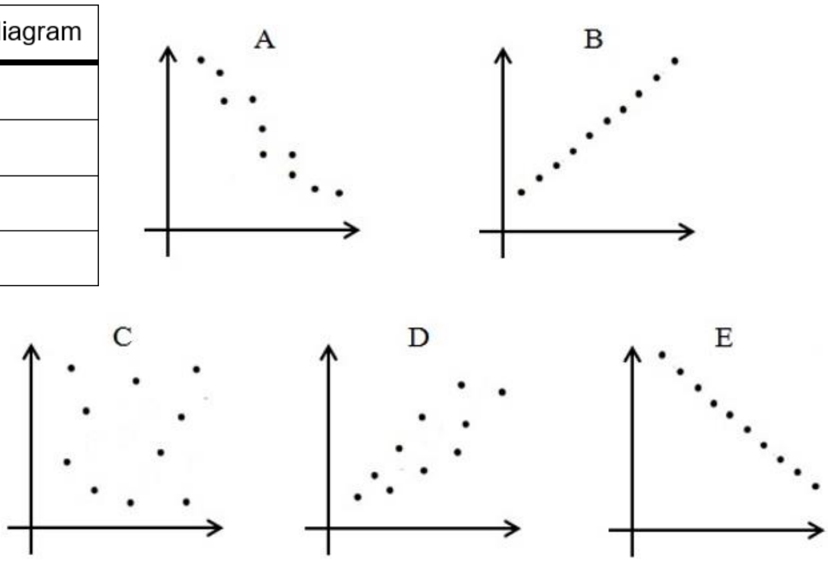
2. [Maximum mark: 6]

There are seven different plants being studied in a biology class. For each plant, x is the diameter of the stem in centimetres and y is the average leaf length in centimetres. Let r be the Pearson's product-moment correlation coefficient.

(a) Write down the possible minimum and maximum values of r. [2]

(b) Copy and complete the following table by noting which scatter diagram A, B, C, D or E corresponds to each value of r. [4]

correlation coefficient r	scatter diagram
-1	
-0.8	
0	
0.5	



(a) $r_{\max} = 1,\; r_{\min} = -1$

(b)

correlation coefficient r	scatter diagram
-1	**E**
-0.8	**A**
0	**C**
0.5	**D**

3. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B)=0.6$ and $\mathrm{P}(A\cup B)=0.7$. Find $\mathrm{P}(A\mid B)$.

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$$

$$0.7=0.3+0.6-\mathrm{P}(A\cap B)$$

$$\mathrm{P}(A\cap B)=0.2$$

$$\mathrm{P}(A\mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{0.2}{0.6}$$

$$\mathrm{P}(A\mid B)=\frac{1}{3}$$

4. [Maximum mark: 5]

Let n and $n+1$ be any two consecutive integers where $n \in \mathbb{Z}$. Hence, prove that the sum of the squares of any two consecutive integers is odd.

n and $n+1$ are any two consecutive integers where $n \in \mathbb{Z}$

$$n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1$$
$$= 2n^2 + 2n + 1$$
$$= 2(n^2 + n) + 1$$

The expression $2(n^2 + n)$ is divisible by 2, so it must be an even number

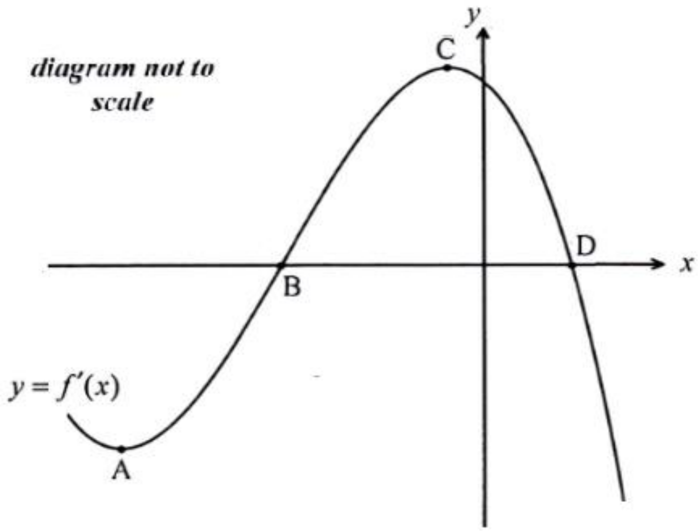
Adding 1 to an even number produces an odd number

Hence, the expression $2(n^2 + n) + 1$ must be an odd number

Therefore, the sum of any two consecutive integers is odd ***Q.E.D.***

5. [Maximum mark: 7]

The diagram shows part of the graph of $y = f'(x)$, the **derivative** of function f. The x-intercepts are at points B and D and there is a minimum at point A and a maximum at point C.



(a) (i) Write down the value of $f'(x)$ at B.

(ii) Hence, verify that the x-coordinate of B is also the x-coordinate of a minimum on the graph of f. [3]

(b) Which of the points A, C or D corresponds to a maximum on the graph of f? [1]

(c) Verify that C corresponds to a point of inflexion on the graph of f. [3]

(a) (i) $f'(x) = 0$ at point B

(ii) As x increases and passes through the x-coordinate of B $f'(x)$ changes from negative to positive.
Hence, the graph of f is decreasing before the x-coordinate of B and increasing after the x-coordinate of B.
Thus, the graph of f has a minimum at the x-coordinate of B.

(b) The graph of f has a maximum at point D

(c) As x increases and passes through C, $f'(x)$ is increasing then decreasing; that is, $f'(x)$ changes from positive to negative at point C.
Thus, point C is an inflexion point on the graph of f.

6. [Maximum mark: 6]

A geometric series has a common ratio of 2^x.

(a) Find the values of x for which the sum to infinity of the series exists. [2]

(b) If the first term of the series is 14 and the sum to infinity is 16, find the value of x. [4]

(a) S_∞ exists if $-1 < r < 1 \Rightarrow |r| < 1$

$|2^x| < 1 \Rightarrow x < 0$

(b) $S_\infty = \dfrac{u_1}{1-r}$

$16 = \dfrac{14}{1-2^x}$

$1-2^x = \dfrac{14}{16} = \dfrac{7}{8}$

$2^x = \dfrac{1}{8}$

$x = -3$

Section B

7. [Maximum mark: 13]

All of the students in a class of 35 must study at least one science – either Biology or Chemistry. Some of the students study both. 25 students study Biology and 15 students study Chemistry.

(a) (i) Find the number of students who study both Biology and Chemistry

(ii) Write down the number of students who study only Biology. [3]

(b) One student is selected at random from the class.

(i) Find the probability that the student studies only one science.

(ii) Given that the student selected studies only one science, find the probability that the student studies Biology. [5]

Let B be the event that a student studies Biology and C be the event that a student studies Chemistry.

(c) Show that B and C are **not** mutually exclusive. [2]

(d) Show that B and C are **not** independent events. [3]

■ **worked solution** ■

(a) (i) $35 = 25 + 15 - n(B \cap C)$

$n(B \cap C) = 5$

5 students study both Biology and Chemistry

(ii) 25 students study Biology; 5 students study both Biology and Chemistry

Thus, 20 students study only Biology

(b) (i) # of students studying only Chemistry $= 15 - 5 = 10$

Thus, $\text{P(one science)} = \dfrac{20+10}{35} = \dfrac{30}{35} = \dfrac{6}{7}$

(ii) $\text{P(Biology} \mid \text{one science)} = \dfrac{\text{P}\left(B \cap \text{one science}\right)}{\text{P}\left(\text{one science}\right)} = \dfrac{\frac{20}{35}}{\frac{6}{7}} = \dfrac{4}{7} \cdot \dfrac{7}{6} = \dfrac{4}{6} = \dfrac{2}{3}$

(c) If B and C are mutually exclusive then $\text{P}(B \cup C) = \text{P}(B) + \text{P}(C)$.

However, $\text{P}(B \cup C) = 1$ and $\text{P}(B) + \text{P}(C) = \dfrac{25}{35} + \dfrac{15}{35} \neq 1$.

Thus, B and C are **not** mutually exclusive.

(d) If B and C are independent events then $\text{P}(B \cap C) = \text{P}(B) \cdot \text{P}(C)$.

However, $\text{P}(B \cap C) = \dfrac{5}{35} = \dfrac{1}{7}$ and $\text{P}(B) \cdot \text{P}(C) = \dfrac{25}{35} \cdot \dfrac{15}{35} = \dfrac{5}{7} \cdot \dfrac{3}{7} \neq \dfrac{1}{7}$

Thus, B and C are **not** independent events.

8. [Maximum mark: 16]

The function f is defined as $f(x)=\dfrac{x+1}{\ln(x+1)}$, $x>0$.

(a) (i) Show that $f'(x)=\dfrac{\ln(x+1)-1}{\left(\ln(x+1)\right)^2}$.

(ii) Find $f''(x)$, writing it as a single rational expression [6]

(b) (i) Find the value of x satisfying the equation $f'(x)=0$.

(ii) Show that this value gives a minimum value for $f(x)$, and determine the minimum value of the function. [7]

(c) Find the x-coordinate of the one point of inflexion on the graph of f. [3]

■ **worked solution** ■

(a) (i) Using the quotient rule.

$$f'(x)=\frac{\left(\ln(x+1)\right)(1)-(x+1)\left(\dfrac{1}{x+1}\right)}{\left(\ln(x+1)\right)^2}$$

So $f'(x)=\dfrac{\ln(x+1)-1}{\left(\ln(x+1)\right)^2}$

(ii) **METHOD 1**

Using the quotient rule.

$$f''(x)=\frac{\dfrac{\left(\ln(x+1)\right)^2}{x+1}-\dfrac{2\ln(x+1)\left(\ln(x+1)-1\right)}{x+1}}{\left(\ln(x+1)\right)^4}$$

$$=\frac{2\ln(x+1)-\left(\ln(x+1)\right)^2}{(x+1)\left(\ln(x+1)\right)^4}$$

$$=\frac{2-\ln(x+1)}{(x+1)\left(\ln(x+1)\right)^3}$$

worked solution for question 8 continues on next page >>

worked solution for question 8 continued

(a) (ii) **METHOD 2**

$$f'(x)=\frac{1}{\ln(x+1)}-\frac{1}{\left(\ln(x+1)\right)^2}$$

$$f''(x)=\frac{-1}{(x+1)\left(\ln(x+1)\right)^2}+\frac{2}{(x+1)\left(\ln(x+1)\right)^3}$$

$$=\frac{2-\ln(x+1)}{(x+1)\left(\ln(x+1)\right)^3}$$

(b) (i) $\ln(x+1)=1$

$x=\mathrm{e}-1$

(ii) **METHOD 1**

Using a first derivative test.

For example, when $x=1$, $f'(x)=\ln 2-1(<0)$.

For example, when $x=2$, $f'(x)=\ln 3-1(>0)$.

Hence, $x=\mathrm{e}-1$ gives a minimum value.

$$f(\mathrm{e}-1)=\frac{\mathrm{e}-1+1}{\ln(\mathrm{e}-1+1)}=\frac{\mathrm{e}}{\ln(\mathrm{e})}=\frac{\mathrm{e}}{1}=\mathrm{e}$$

Thus, the minimum value is e.

METHOD 2

Using the second derivative test.

$$f''(\mathrm{e}-1)=\frac{1}{\mathrm{e}}>0$$

Hence, $x=\mathrm{e}-1$ gives a minimum value.

$$f(\mathrm{e}-1)=\frac{\mathrm{e}-1+1}{\ln(\mathrm{e}-1+1)}=\frac{\mathrm{e}}{\ln(\mathrm{e})}=\frac{\mathrm{e}}{1}=\mathrm{e}$$

Thus, the minimum value is e.

(c) $2-\ln(x+1)=0$

$\ln(x+1)=2$

$x=\mathrm{e}^2-1$

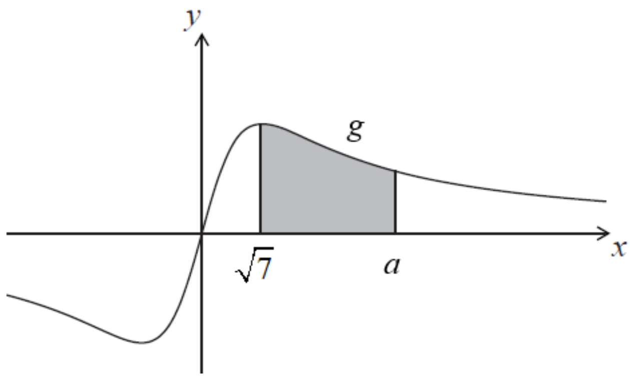
9. [Maximum mark: 16]

The function g is defined by $g(x) = \dfrac{3x}{x^2+7}$.

(a) Show that $g'(x) = \dfrac{21-3x^2}{\left(x^2+7\right)^2}$. [5]

(b) Find $\displaystyle\int \frac{3x}{x^2+7}\,\mathrm{d}x$. [4]

The diagram below shows a portion of the graph of g.



(c) The shaded region is enclosed by the graph of g, the x-axis, and the lines $x=\sqrt{7}$ and $x=a$ such that $a>\sqrt{7}$. This region has an area of $\ln 8$. Find the value of a. [7]

■ worked solution ■

(a) Applying the quotient rule:

$$g(x) = \frac{\left(x^2+7\right)(3)-(2x)(3x)}{\left(x^2+7\right)^2} = \frac{3x^2+21-6x^2}{\left(x^2+7\right)^2}$$

Thus, $g'(x) = \dfrac{21-3x^2}{\left(x^2+7\right)^2}$ ***Q.E.D.***

(b) $\displaystyle\int \frac{3x}{x^2+7}\,\mathrm{d}x = \int \frac{1}{x^2+7}3x\,\mathrm{d}x$ let $u = x^2+7$, then $\mathrm{d}u = 2x\,\mathrm{d}x \Rightarrow \dfrac{3}{2}\mathrm{d}u = 3x\,\mathrm{d}x$

Substituting gives $\displaystyle\int \frac{3x}{x^2+7}\,\mathrm{d}x = \int \frac{1}{u}\cdot\frac{3}{2}\,\mathrm{d}u = \frac{3}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{3}{2}\ln u$

Thus, $\displaystyle\int \frac{3x}{x^2+7}\,\mathrm{d}x = \frac{3}{2}\ln\left(x^2+7\right)+C$

worked solution for question 9 continues on next page >>

worked solution for question 9 continued

(c) area $=\int_{\sqrt{7}}^{a}\frac{3x}{x^2+7}\,\mathrm{d}x=\ln 8$

$$\frac{3}{2}\ln\left(x^2+7\right)\Big]_{\sqrt{7}}^{a}=\ln 8$$

$$\frac{3}{2}\left[\ln\left(a^2+7\right)-\ln\left(7+7\right)\right]=\ln 8$$

$$\ln\frac{a^2+7}{14}=\frac{2}{3}\ln\left(2^3\right)$$

$$\ln\frac{a^2+7}{14}=\ln\left[\left(2^3\right)^{\frac{2}{3}}\right]$$

$$\ln\frac{a^2+7}{14}=\ln 4$$

$$\frac{a^2+7}{14}=4 \;\Rightarrow\; a^2+7=56 \;\Rightarrow\; a^2=49$$

$a=\pm 7$; but given that $a>\sqrt{7}$

Thus, $a=7$